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Spinor Fields with Zero Mass in Unbounded Isotropic Media

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The Dirac equation for massless fields in unbounded media has solutions similar to the focus wave mode solutions of Maxwell's equations leading to infinite dynamical invariants. We define the splash wave mode solutions as a weighted superposition of the focus wave modes, and discuss the conditions to be fulfilled by the weight functions to make the dynamical invariants bounded. We leave open the physical interpretation of these solutions.

1. INTRODUCTION

We show that the Dirac equation for massless fields is unbounded isotropic media has focus wave and splash wave solutions.

Using the cylindrical coordinates (r, φ, z) and the natural system of units $(\hbar = c = 1)$, we have the Dirac equation in the form

$$\hat{\partial}\Psi \equiv \left(\Gamma^r \partial_r + \frac{1}{r}\Gamma^\varphi \partial_\varphi + \Gamma^z + \Gamma^0 \partial_t\right)\Psi = 0 \tag{1}$$

where ∂_r, ∂_φ, ∂_z, and ∂_t are the derivatives with respect to space and time and the matrices Γ are

$$\Gamma^r = \begin{vmatrix} 0 & \sigma_r \\ -\sigma_r & 0 \end{vmatrix}, \qquad \Gamma^\varphi = \begin{vmatrix} 0 & \sigma_\varphi \\ -\sigma_\varphi & 0 \end{vmatrix}, \qquad \Gamma^z = \begin{vmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{vmatrix},$$

$$\Gamma^0 = \begin{vmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{vmatrix}, \qquad \Gamma^5 = \begin{vmatrix} -\sigma_0 & 0 \\ 0 & \sigma_0 \end{vmatrix} \tag{2}$$

with $(i = \sqrt{-1})$

$$\sigma_r = \begin{vmatrix} 0 & e^{-i\varphi} \\ e^{i\varphi} & 0 \end{vmatrix}, \qquad \sigma_\varphi = \begin{vmatrix} 0 & -ie^{-i\varphi} \\ ie^{i\varphi} & 0 \end{vmatrix}, \qquad \sigma_z = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \tag{2'}$$

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σ_0 is the 2×2 identity matrix. The matrices Γ and σ satisfy the commutation relations of the Dirac and Pauli algebras, respectively; one has

$$\sigma_r\sigma_\varphi = i\sigma_z, \qquad \sigma_\varphi\sigma_z = i\sigma_r, \qquad \sigma_z\sigma_r = i\sigma_\varphi \tag{3}$$

and

$$\partial_\varphi\sigma_r = \sigma_\varphi, \qquad \partial_\varphi\sigma_\varphi = -\sigma_r \tag{3'}$$

We note that $\Psi^\dagger$, $\bar\Psi = \Psi^\dagger\Gamma^0$, and $\Psi^c = c\Gamma^0\Psi^*$ are the hermitian conjugate, adjoint, and charge conjugate fields, respectively. The asterisk denotes complex conjugation and one has

$$e^{-1}\Gamma C = \Gamma^T, \qquad e = i\begin{vmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{vmatrix}, \qquad \sigma_2 = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix} \tag{4}$$

The equation for the adjoint field is

$$\hat\partial^\dagger\bar\Psi \equiv (\partial_r\bar\Psi)\Gamma^r + \frac{1}{r}(\partial_\varphi\bar\Psi)\Gamma^\varphi + (\partial_z\bar\Psi)\Gamma^z + (\partial_t\bar\Psi)\Gamma^0 = 0 \tag{1'}$$

Using (2), (2′) and (3), (3′) we get easily the relation

$$\hat\partial^2 = \mathbb{D} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}$$

so that every component of Ψ is a solution of the wave equation $\mathbb{D}u = 0$. Let us take Ψ in the form

$$\Psi = \begin{pmatrix}\Phi \\ X\end{pmatrix}, \qquad \Phi = \begin{pmatrix}\varphi_1 \\ \varphi_2\end{pmatrix}, \qquad X = \begin{pmatrix}\chi^{\dot 1} \\ \chi^{\dot 2}\end{pmatrix} \tag{5}$$

Then equation (1) reduces to a system of two equations:

$$\begin{aligned} \partial_\alpha^\beta\varphi_\beta &= \left(\sigma_r\partial_r + \frac{1}{r}\sigma_\varphi\partial_\varphi + \sigma_z\partial_z + \partial_t\right)\Phi = 0, \qquad \alpha, \beta = 1, 2 \\ \partial_\beta^{\dot\alpha}\chi^{\dot\beta} &\equiv \left(\sigma_r\partial_r + \frac{1}{r}\sigma_\varphi\partial_\varphi + \sigma_z\partial_z - \partial_t\right)X = 0, \qquad \dot\alpha, \dot\beta = 1, 2 \end{aligned} \tag{6}$$

From now on we only consider the solutions of (6) corresponding to waves propagating along $0z$.

2. FOCUS WAVE MODE SOLUTIONS

2.1. First Kind of Focus Wave Modes

It is easy to check that equations (6) have the solutions (Hillion, 1987)

$$\begin{aligned} \varphi_{1,n} &= \varphi_{n+1}, \qquad \chi_m^{\dot 1} = i\chi_m \\ \varphi_{2,n} &= -i\varphi_n \qquad \chi_m^{\dot 2} = -\chi_{m+1} \end{aligned} \tag{7}$$

with

$$\varphi_n = \frac{r^n}{(a-i\xi)^{n+1}} \exp\left(-\frac{kr^2}{a-i\xi}\right) \exp[-i(k\bar{\xi}+n\varphi)]$$
$$\chi_m = \frac{r^m}{(b+i\xi)^{m+1}} \exp\left(-\frac{qr^2}{b+i\xi}\right) \exp[i(q\bar{\xi}+m\varphi)] \tag{7'}$$

where we used the variables $\xi = z\text{-}ct$ and $\bar{\xi} = z+ct$ (with $c=1$). The parameters (a, k, n) and (b, q, m) are arbitrary, but to obtain bounded solutions we assume that a, b, k, and q are some positive real scalars, while m and n are positive half-integers.

Substituting (7) into (5) gives the solutions Ψ_{nm} of Eq. (1) corresponding to spinor waves propagating along $0z$.

Let us now consider instead of (6) the Proca equations (Hillion and Quinnez 1986a)

$$\partial^\beta_\alpha \varphi^\gamma_\beta = 0 \qquad \partial^{\dot\alpha}_\beta \chi^{\dot\beta}_{\dot\gamma} = 0 \tag{8}$$

where φ^γ_β and $\chi^{\dot\beta}_{\dot\gamma}$ are traceless second-rank spinors. A look at (6) and (8) supplies at once the solutions:

$$\begin{aligned}
&\varphi^1_{1,n} = \varphi_n, &&\varphi^2_{1,n} = -i\varphi_{n+1}, &&\chi^{\dot 1}_{\dot 2,m} = i\chi_m, &&\chi^{\dot 1}_{\dot 2,m} = -\chi_{m-1}\\
&\varphi^1_{2,n} = -i\varphi_{n-1}, &&\varphi^2_{2,n} = -\varphi_n, &&\chi^{\dot 2}_{\dot 1,m} = -\chi_{m+1}, &&\chi^{\dot 2}_{\dot 2,m} = -i\chi_m
\end{aligned} \tag{8'}$$

Using the well-known relation between self-dual tensors and traceless second-rank spinors, we get the relations

$$\begin{aligned}
&\Lambda_r + i\Lambda_\varphi = e^{i\varphi}\varphi^2_1\\
&\Lambda_r - i\Lambda_\varphi = ie^{-i\varphi}\varphi^1_2 \qquad \Lambda = \varepsilon^{1/2}E + i\mu^{1/2}H\\
&\Lambda_3 = \tfrac{1}{2}(\varphi^1_1 - \varphi^2_2)
\end{aligned} \tag{9}$$

where E and H are the electric and magnetic fields, ε is the permittivity, and μ, is the permeability. One has similar expressions with $\chi^{\dot\alpha}_{\dot\beta}$, leading to an electromagnetic field with opposite polarization.

Substituting (8′) into (9) and assuming that n is a positive integer, we obtain the focus wave mode solutions (Brittingham, 1985) of Maxwell's equations (Hillion, 1986). This justifies the name given to the solutions (7).

Remark. Let us consider a symmetric second-rank spinor $\varphi_{\alpha\beta}$ instead of a traceless spinor φ^β_α, we easily check the following relations:

$$\begin{aligned}
&\varphi_{11} \equiv \varphi_{\nu+1}(2k) = (a-i\xi)\varphi^2_n(k) \equiv (a-i\xi)(\varphi_1)^2\\
&\varphi_{22} \equiv \varphi_{\nu-1}(2k) = (a-i\xi)\varphi^2_{n-1}(k) \equiv (a-i\xi)(\varphi_2)^2\\
&\varphi_{12} = \varphi_{21} \equiv \varphi_\nu(2k) = (a-i\xi)\varphi_n(k)\varphi_{n-1}(k) \equiv (a-i\xi)\varphi_1\varphi_2
\end{aligned}$$

with $\nu = 2n - 1$. These kinds of relations are at the base of de Broglie's method of fusion in his theory of light (de Broglie, 1940).

2.2. Relativistic Covariance

Let us consider the Lorentz transformation:

$$x' = x, \qquad y' = y, \qquad z' = \frac{z - vt}{(1-\beta^2)^{1/2}}, \qquad t' = \frac{t - \beta z/c}{(1-\beta^2)^{1/2}}, \qquad \beta = \frac{v}{c} \tag{10}$$

From (10) we get

$$(\xi, k, q, a, b) \mapsto (\xi', q', k', a', b') = \left(\frac{1+\beta}{1-\beta}\right)^{1/2} (\xi, k, q, a, b) \tag{11a}$$

$$\bar{\xi} \mapsto \bar{\xi}' = \left(\frac{1-\beta}{1+\beta}\right)^{1/2} \bar{\xi} \tag{11b}$$

Since r and φ are transverse coordinates, they are invariant. So if we write φ_n in the form

$$\varphi_n = \frac{(r/r_0)^n}{(1 - i\xi/kr_0^2)^{n+1}} \exp\left[-\frac{(r/r_0)^2}{1 - i\xi/kr_0^2}\right] \exp(-i(k\bar{\xi} + n\varphi)]$$

and similarly χ_m, one sees at once, using (7), (7'), and (10) that the Lorentz transformation (10) leads to

$$\begin{aligned} \Phi_n \mapsto \Phi_n' = U\Phi_n, \qquad & U = \begin{vmatrix} [(1+\beta)/(1-\beta)]^{1/2} & 0 \\ 0 & 1 \end{vmatrix} \\ \chi_m \mapsto \chi_m' = V\chi_m, \qquad & V = \begin{vmatrix} 1 & 0 \\ 0 & [(1+\beta)/(1-\beta)]^{1/2} \end{vmatrix} \end{aligned} \tag{12}$$

that is, according to (5),

$$\Psi_{nm} \mapsto \Psi'_{nm} = S'\Psi_{nm}, \qquad S = \begin{vmatrix} U & 0 \\ 0 & V \end{vmatrix} \tag{13}$$

A simple calculation gives

$$S^{-1}\Gamma^{r,\varphi}S = \Gamma^{r,\varphi}, \qquad S^{-1}(\Gamma^0 \pm \Gamma^z)S = \left(\frac{1 \mp \beta}{1 \pm \beta}\right)^{1/2} (\Gamma^0 \pm \Gamma^z) \tag{14}$$

According to (13) and (14), equation (1) is covariant under the Lorentz transformation (10).

2.3. Dynamical Invariants

The Dirac equation (1) and its adjoint equation (1′) can be obtained from the Lagrangian

$$\mathscr{L}=\tfrac{1}{2}[\bar{\Psi}\hat{\partial}\Psi-(\hat{\partial}^{\dagger}\bar{\Psi})\Psi] \tag{15}$$

So, assuming real all the parameters in the solutions Ψ_{nm}, the components $T^{0,\cdot}$ of the energy-momentum tensor, where the dot stands for the indices r, φ, z, 0, take the form

$$\begin{aligned}
T^{0,r}_{n,m} &= \frac{i}{2}(\Psi^{\dagger}_{n,m}\partial_r\Psi_{n,m}-\partial_r\Psi^{\dagger}_{n,m}\Psi_{n,m}) \\
&= 2r\xi\left(\frac{k}{a^2+k^2}|\Phi_n|^2-\frac{q}{b^2+q^2}|X_m|^2\right) \\
T^{p,\varphi}_{n,m} &= \frac{i}{2r}(\Psi^{\dagger}_{n,m}\partial_{\varphi}\Psi_{n,m}-\partial_{\varphi}\Psi^{\dagger}_{n,m}\Psi_{n,m}) \\
&= \frac{1}{r}(n|\Phi_n|^2+|\varphi_{1,n}|^2-m|X_m|^2-|\chi^{1}_{m}|^2) \\
T^{0,0}_{n,m}+T^{0,z}_{n,m} &= \frac{i}{2}(\Psi^{\dagger}_{n,m}\partial_{\bar{\xi}}\Psi_{n,m}-\partial_{\bar{\xi}}\Psi^{\dagger}_{n,m}\Psi_{n,m}) \\
&= k|\Phi_n|^2-q|X_m|^2 \\
T^{0,0}_{n,m}-T^{0,z}_{n,m} &= -\frac{i}{2}(\Psi^{\dagger}_{n,m}\partial_{\xi}\Psi_{n,m}-\partial_{\xi}\Psi^{\dagger}_{n,m}\Psi_{n,m}) \\
&= \frac{a}{a^2+\xi^2}\left(|\varphi_{1,n}|^2+\left(n+1-\frac{kr^2}{a}\frac{a^2-\xi^2}{a^2+\xi^2}\right)|\Phi_n|^2\right) \\
&\quad -\frac{b}{b^2+\xi^2}\left[|\chi^{i}_{m}|^2 r\left(m+1-\frac{qr^2}{b}\frac{b^2-\xi^2}{b^2+\xi^2}\right)|X_m|^2\right]
\end{aligned} \tag{16}$$

with

$$\begin{aligned}
|\Phi_n|^2 &= \left(1+\frac{r^2}{a^2+\xi^2}\right)|\varphi_n|^2 \\
|\varphi_n|^2 &= \frac{r^{2n}}{(a^2+\xi^2)^{n+1}}\exp\left(-\frac{2akr^2}{a^2+\xi^2}\right)
\end{aligned} \tag{17}$$

and similar expressions for $|X_m|^2$ and $|\chi_m|^2$.

For the component $S^{0,r\varphi}_{nm}$ of the spin tensor we get

$$S^{0,r\varphi}_{n,m}=\frac{1}{2i}(\Psi^{\dagger}_{n,m}\Gamma^{r}\Gamma^{\varphi}\Psi_{n,m})$$
$$=\frac{1}{2}\left(1-\frac{r^2}{a^2+\xi^2}\right)|\varphi_n|^2-\frac{1}{2}\left(1-\frac{r^2}{b^2+\xi^2}\right)|\chi_m|^2 \tag{18}$$

The physical quantities of interest are the volume integrals of these densities, that is, the energy-momentum four-vector:

$$P^{\cdot}_{n,m}=\int_{-\infty}^{+\infty}d\xi\int_0^{\infty}r\,dr\int_0^{2\pi}T^{0,\cdot}_{n,m}\,d\varphi \tag{19}$$

and the $0z$ component of the spin vector:

$$S^{\xi}_{n,m}=\int_{-\infty}^{+\infty}d\xi\int_0^{\infty}r\,dr\int_0^{2\pi}S^{0,r\varphi}_{n,m}\,d\varphi \tag{19'}$$

In Appendix A, we prove the following results:

1. $P^{r}_{n,m}=0$, in agreement with the fact that $\Psi_{n,m}$ represents a wave propagating along $0z$.
2. $P^{\varphi}_{n,m}$, $P^{z}_{n,m}$, $P^{0}_{n,m}$, and $S^{z}_{n,m}$ are infinite, a not surprising result, since we considered an unbounded medium without any source or sink. Similarly for the focus wave mode solutions of Maxwell's equations, the electromagnetic energy is infinite (Wu, and Lehmann, 1985).

Nevertheless, for the self-conjugate fields $\Psi_{n,m}=\Psi^{c}_{n,m}$, one has, according to (4), $|X_m|^2=|\Phi_n|^2$, which implies $m=n$, $b=a$, $q=k$. Then all the components are zero. Such a field $\Psi_{n,m}$ made up of a doublet of charge conjugate spinors carries no mass, energy, momentum, charge or spin.

2.4. Second Kind of Focus Wave Mode

We note $u=\{r,\varphi,\bar{\xi}\}$ and let Ψ_f be

$$\Psi_f(u,\xi)=\int_{-\infty}^{+\infty}f(\xi-s)\Psi(u,s)\,ds \tag{20}$$

where f is a differentiable function null at the infinity such that the integral (20) exists. Then it is easy to check that if Ψ is a solution of equation (1), Ψ_f is also a solution of equation (1).

Let us consider, for instance, the Gabor (1946) transformation. Starting with the following Gaussian function shifted in direct and Fourier transform spaces:

$$g(s,\xi;w)=\frac{1}{(2\pi\sigma^2)^{1/4}}\exp\left[-\frac{(s-\xi)^2}{4\sigma^2}+iw\left(s-\frac{\xi}{2}\right)\right] \tag{21}$$

where σ and w are positive scalars, the Gabor transform Ψ_g of Ψ is defined by the relation

$$\Psi_g(u, \xi; w) = \int_{-\infty}^{+\infty} g(s, \xi; w)\Psi(u, s)\, ds \tag{22}$$

In (22), it is assumed that every component of Ψ is an arbitrary, complex-valued, square-integrable function (with respect to the variable s). The inverse transform is (Gabor, 1946; Helstrom, 1966)

$$\Psi(u, s) = \int \int_{-\infty}^{+\infty} g^*(s, \xi; w)\Psi_g(u, \xi; w)\, d\xi\, dw \tag{23}$$

and one has

$$\int \int_{-\infty}^{+\infty} |\Psi_g|^2\, d\xi\, dw = \int_{-\infty}^{+\infty} |\Psi|^2\, ds \tag{24}$$

This last relation proceeds from the definition of $|\Psi|^2$ as $\sum_{\alpha=1}^{4}|\varphi_\alpha|^2$ and from the fact that (24) holds for every component φ_α (Helstrom, 1966).

We now define the spinor fields $\Omega(u, \xi; w)$ through the relation

$$\Omega(u, \xi; w) = e^{-i\xi w/2}\Psi_g(u, \xi; w) \tag{25}$$

According to (20)–(22), $\Omega(u, \xi; w)$ is a solution of the Dirac equation (1). Substituting into (22) the solutions $\Psi_{n,m}(u, s)$ given by (5) and (7) leads to a second kind of focus wave mode solution $\Omega_{n,m}(u, \xi; w)$. We have not proved that with $\Omega_{n,m}$ the conditions 1 and 2 of the last section on the dynamical invariants are still satisfied, but the relation (24) makes this result plausible.

3. SPLASH WAVE MODE SOLUTIONS

It it not a drawback per se that the solutions $\Psi_{n,m}$ and $\Omega_{n,m}$ have infinite energy. Plane wave solutions also share this property. But it is interesting (Ziolkowski, 1985) to look for solutions $\hat{\Psi}_{n,m}$ and $\hat{\Omega}_{n,m}$, which we call splash wave modes (Ziolkowski, 1985), such as the quantities $P^{\cdot}_{n,m}$ and $S^{z}_{n,m}$ are finite. We define $\hat{\Psi}_{n,m}$ and $\hat{\Omega}_{n,m}$ as weighted superposition of $\Psi_{n,m}$ and $\Omega_{n,m}$, respectively.

Denoting $\Phi_n(k)$, $X_m(q)$ the solutions (7), we get

$$\hat{\Psi}_{n,m} = \begin{pmatrix} \hat{\Phi}_n \\ \hat{X}_m \end{pmatrix} = \int_0^{\infty} \begin{pmatrix} F_1(k)\, dk & 0 \\ 0 & F_2(q)\, dq \end{pmatrix} \begin{pmatrix} \Phi_n(k) \\ X_m(q) \end{pmatrix} \tag{26}$$

where F_1 and F_2 are suitable weight functions.

Let us introduce the Laplace-like transforms

$$G_1(p_a)=\int_0^\infty F_1(k)\,e^{-kp_a}\,dk,\qquad G_2(p_b)=\int_0^\infty F_2(q)\,e^{-qp_b}\,dq \tag{27}$$

with

$$p_a=i\bar{\xi}+r^2/(a-i\xi),\qquad p_b=-i\bar{\xi}+r^2/(b+i\xi) \tag{27'}$$

Then the solutions $\hat{\Phi}_n$, $\hat{X}_m$ take the form

$$\begin{aligned}\hat{\Phi}_n&=\frac{r^n\,e^{-in\varphi}}{(a-i\xi)^{n+1}}\,G_1(p_a)\begin{pmatrix} r\,e^{-i\varphi}/(a-i\xi)\\ -i\end{pmatrix}\\ \hat{X}_m&=\frac{r^m\,e^{im\varphi}}{(b+i\xi)^{m+1}}\,G_2(p_b)\begin{pmatrix} i\\ -r\,e^{i\varphi}/(b+i\xi)\end{pmatrix}\end{aligned} \tag{28}$$

Using in the expressions (16) and (18) $\hat{\Phi}_n$ and $\hat{X}_m$ instead of Φ_n and X_m gives the components $\hat{T}^{0,\bullet}_{n,m}$ of the energy-momentrum tensor and the components $\hat{S}^{0,r\varphi}_{nm}$ of the spin tensor. In Appendix B, we prove that $\hat{P}^{\bullet}_{n,m}$ and $\hat{S}^{z}_{nm}$ are finite provided that the integrals

$$Q_{1,s_1}=\int_0^\infty dk\,\frac{|F_1(k)|^2}{k^{s_1}},\qquad Q_{2,s_2}=\int_0^\infty dq\,\frac{|F_2(q)|^2}{q^{s_2}} \tag{29}$$

are bounded for

$$\begin{aligned}s_1&=n-\tfrac{3}{2},\quad n-\tfrac{1}{2},\quad n,\quad n+\tfrac{1}{2},\quad n+1\\ s_2&=m-\tfrac{3}{2},\quad m-1,\quad m-\tfrac{1}{2},\quad m,\quad m+\tfrac{1}{2},\quad m+1\end{aligned}$$

These rather mild conditions are easy to satisfy, for instance, with weight functions such as $k^l\,e^{-k\beta}$ and $k^lJ_\nu(k\beta)$, where μ is positive and J_ν denotes the usual Bessel function. A qualitative discussion of $\hat{\Phi}_n$ can be found in Hillion (1987) for this last weight function.

One should have similar results for $\hat{\Omega}_{nm}$, but we have not made explicit computations. Let us remark that one could also define another kind of solution $\hat{\hat{\Omega}}_{n,m}$ by weighting the parameter w:

$$\hat{\hat{\Omega}}_{n,m}(u,\xi;v)=\int_{-\infty}^{+\infty}F(vw)\Omega_{n,m}(u,\xi;w)\,dw \tag{30}$$

We have not checked these solutions.

4. PARAXIAL APPROXIMATION OF THE DIRAC EQUATION

Using the variables ξ and $\bar{\xi}$ we find that the wave equation $\mathbb{D}\psi=0$ becomes

$$\left(\Delta_\perp+4\frac{\partial^2}{\partial\xi\,\partial\bar{\xi}}\right)\psi=0 \tag{31}$$

where $\Delta_\perp$ is the transverse Laplacian. The paraxial approximation of (31) is

$$\left(\Delta_\perp + 2\frac{\partial^2}{\partial z\,\partial\bar{\xi}}\right)\tilde{\psi} = 0 \tag{32}$$

since, for the solutions $\tilde{\psi}(x_\perp, z)\,e^{ik\bar{\xi}}$, (32) leads to the parabolic equation

$$\left(\Delta_\perp + 2ik\frac{\partial}{\partial z}\right)\tilde{\psi} = 0 \tag{32'}$$

satisfied by the paraxial approximation of the scalar fields. A look at (31) and (32) shows that if $\psi(x_\perp, \xi, \bar{\xi})$ is a solution of (31), then $\tilde{\psi} \equiv \psi(x_\perp, 2z, \bar{\xi})$ is a solution of (32). In particular, the focus wave solutions ψ_n of Eq. (31) lead to the solutions $\tilde{\psi}_n$ used to discuss the propagation of Gaussian light beams.

With the variables ξ and $\bar{\xi}$ the Dirac equation (1) becomes

$$(\hat{\partial}_\perp + 2\Gamma^{\xi}\partial_{\xi} + 2\Gamma^{\bar{\xi}}\partial_{\bar{\xi}})\Psi = 0 \tag{33}$$

$\hat{\partial}_\perp$ is the transverse part of the operator $\hat{\partial}$ and one has

$$\Gamma^{\xi} = \tfrac{1}{2}(\Gamma^z - \Gamma^0), \qquad \Gamma^{\bar{\xi}} = \tfrac{1}{2}(\Gamma^z + \Gamma^0) \tag{33'}$$

These matrices satisfy the relations

$$(\Gamma^{\xi})^2 = (\Gamma^{\bar{\xi}})^2 = 0, \qquad \Gamma^{\xi}\Gamma^{\bar{\xi}} + \Gamma^{\bar{\xi}}\Gamma^{\xi} = \tfrac{1}{2} \tag{34}$$

Substituting in (32) ∂_z by $2\partial_{\xi}$ gives the paraxial approximation of the Dirac equation:

$$(\hat{\partial}_\perp + \Gamma^{\xi}\partial_z + 2\Gamma^{\bar{\xi}}\partial_{\bar{\xi}})\tilde{\Psi} = 0 \tag{35}$$

since it is easy to check that each solution of (35) is also a solution of (32).

So the solutions $\Psi(x_\perp, \xi, \bar{\xi})$ of (33) supply the solutions $\tilde{\Psi}(x_\perp, 2z, \bar{\xi})$ of (35) and in particular the paraxial spinor fields $\tilde{\Psi}_{nm}$, $\tilde{\Omega}_{nm}$, $\tilde{\bar{\Psi}}_{nm}$ $\tilde{\bar{\Omega}}_{nm}$, obtained from the focus wave modes and from the splash wave modes. Note that the paraxial waves have no longitudinal structure, so that the problem of the infinities disappears.

Using cylindrical coordinates together with the representation (2), (2′) of the Dirac matrices, we find that equation (35) becomes

$$\begin{aligned} &2\partial_{\bar{\xi}}\tilde{\varphi}_1 + e^{i\varphi}\left(\partial_r - \frac{i}{r}\partial_{\varphi}\right)\tilde{\varphi}_2 = 0 \\ &e^{i\varphi}\left(\partial_r + \frac{1}{r}\partial_{\varphi}\right)\tilde{\varphi}_1 - \partial_z\tilde{\varphi}_2 = 0 \end{aligned} \tag{36}$$

with a similar system for the spinor $\tilde{X}$.

This equation is reminiscent of the equation for the TE, TM electromagnetic modes in a cylindrical waveguide (Hillion and Quinnez, 1986a,b). It could also supply the basis for a geometrical optics approximation of the neutrino field.

Remark 1. Using $\tilde{\varphi}$ instead of φ in the expressions (8′), (9) leads to the paraxial approximation of the electromagnetic field. In particular, for $n = 0$, this gives the Gaussian solutions previously discussed by some authors (Lax *et al.*, 1975; Davis, 1979) with a minor difference. Here we get a circularly polarized field, while they obtained a linearly polarized one.

Remark 2. There exists between the solutions $\tilde{\psi}$ of equation (32′) and the solutions of the Laplace equation $(\Delta_\perp + \partial_z^2)\psi_2 = 0$ the very simply integral relation

$$\tilde{\psi}(x_\perp, z) = \frac{1}{(\pi z)^{1/2}} \int_0^\infty \left[\exp\left(-\frac{iks^2}{2z}\right)\right] \psi_L(x_\perp, s)\, ds \tag{37}$$

provided that $\partial_z\psi_L = 0$ at $z = 0$. This result can be obtained by a direct calculation or by taking the Laplace transform (with respect to z) of both partial differential equations. In this case one has to use the relation between $f(p)$ and $f(\sqrt{p})$, where p denotes the symbolic variable (Van der Pol and Bremmer, 1959).

For instance, with $\psi_L = I_m(k_0 r)\, e^{im\varphi} \cos k_0 z$ we get

$$\tilde{\psi} = I_m(k_0 r) \exp(im\varphi) \exp\left(i\frac{k_0^2}{k}\frac{z}{2}\right) \times \left[\operatorname{erfc}\left(k_0\left(\frac{iz}{2\sqrt{k}}\right)^{1/2}\right) + \operatorname{erfc}\left(-k_0\left(\frac{iz}{2k}\right)^{1/2}\right)\right]$$

where I_m is the usual modified Bessel function, and erfc is the complementary error function.

5. DISCUSSION

The Dirac equation for massless fields in unbounded isotropic media is very rich in solutions. The solutions that we obtained here correspond to waves with transverse and longitudinal structures propagating along $0z$. The question is whether these solutions are supported by some physical process. One can imagine them as excitations of the vacuum or of a hypothetical ether (Dirac, 1951). In particular, the self-conjugate solutions could give birth to excitations undetectable by ordinary means. If these solutions have a physical existence we would still have to find the meaning of the various parameters characterizing the solutions.

APPENDIX A

We prove here the result given in Sections 2 and 3 about $P^{\bullet}_{n,m}$ and $S^z_{n,m}$. Of course one only has to make computations with the half-spinor Φ_n, since X_m gives similar results.

Let μ be a positive integer and α be a positive scalar; one has

$$\int_0^\infty x^\mu e^{-\alpha x^2}\,dx = \frac{\lambda_\mu}{\alpha^{(\mu+1)/2}} \begin{cases} \lambda_\mu = \dfrac{1\cdot 3\cdot 5 - (\mu-1)}{2}\left(\dfrac{\pi}{2\mu}\right)^{1/2} & \text{for} \quad \mu = 2n \\ \lambda_\mu = \dfrac{1}{2}\Gamma\left(\dfrac{\mu+1}{2}\right) & \text{for} \quad \mu = 2n+1 \end{cases} \tag{A1}$$

Γ is the gamma function.

Let n be a positive integer or half-integer; we write

$$\Phi_n = \frac{r^n}{(a-i\xi)^{n+1}} \exp\left(-\frac{kr^2}{a-i\xi}\right) \exp[-i(k\bar{\xi}+n\varphi)] \begin{pmatrix} re^{-i\varphi}/(a-i\xi) \\ -i \end{pmatrix} \tag{A2}$$

and we consider the following quantities, where ν_1 and ν_2 are arbitrary scalars:

$$\Phi^2_{m;\nu_1,\nu_2} = \frac{r^{2n}}{(a^2+\xi^2)^{n+1}}\left(\nu_1 + \nu_2\frac{r^2}{a^2+\xi^2}\right)\exp\left(-\frac{2akr^2}{a^2+\xi^2}\right) \tag{A3}$$

The integrals

$$2\pi\int_{-\infty}^{+\infty} d\xi \int_0^\infty r\,dr\,\Phi^2_{n;\nu_1,\nu_2}, \qquad 2\pi\int_{-\infty}^{+\infty} d\xi\int_0^\infty dr\,\Phi^2_{m;\nu_1,\nu_2}$$

become, according to (A1) and (A3), for $2n = 2s-1$, s a positive integer,

$$2\pi\int_{-\infty}^{\infty} d\xi\int_0^\infty r\,dr\,\Phi^2_{n;\nu_1,\nu_2} = 2\pi\int_{-\infty}^{+\infty} d\xi\left(\frac{\nu_1}{(2ak)^{n+1}}\lambda_{2n+1} + \frac{\nu_2}{(2ak)^{n+2}}\lambda_{2n+2}\right) \tag{A4}$$

$$2\pi\int_{-\infty}^{\infty} d\xi\int_0^{-\infty} dr\,\Phi^2_{n;\nu_1,\nu_2} = 2\pi\int_{-\infty}^{+\infty}\frac{d\xi}{(a^2+\xi^2)^{1/2}} \times\left(\frac{\nu_1}{(2ak)^{n+1/2}}\lambda_{2n} - \frac{\nu_2}{(2ak)^{n+3/2}}\lambda_{2n+1}\right) \tag{A5}$$

The integrals (A4) and (A5) are not bounded.

Let us call $P^{\boldsymbol{\cdot}}_n$ and S^z_n the parts of the components of the energy-momentum four-vector and of the spin vector depending on Φ_n. According to (16)-(19), (19′), we get

$$P^r_n = k\pi\int_{-\infty}^{+\infty}\xi\,d\xi\int_0^\infty dr\,\Phi^2_{n+1;1,1}, \qquad P^\varphi_n = 2\pi\int_{-\infty}^{+\infty} d\xi\int_0^\infty dr\,\Phi^2_{n;n,n+1}$$

$$P^0_n + P^z_n = 2k\pi\int_{-\infty}^{+\infty} d\xi\int_0^\infty r\,dr\,\Phi^2_{n;1,1}, \qquad S^z_n = \pi\int_{-\infty}^{+\infty} d\xi\int_0^\infty r\,dr\,\Phi^2_{n;1,-1} \tag{A6}$$

P_n^r is the expression (A5) with $\xi(a^2+\xi^2)^{-1/2}$ as integrand instead of $(a^2+\xi^2)^{-1/2}$. So $P_n^r=0$.

The other three quantities are not finite, according to (A4) and (A5). We still have to discuss $P_n^0-P_n^z$, which is a bit more intricate. We have

$$P_n^0-P_n^z=2\pi k\int_{-\infty}^{+\infty}d\xi\int_0^\infty r\,dr\,\Phi^2_{n+1;1,1}-4\pi a^2k\int_{-\infty}^{+\infty}\frac{d\xi}{a^2+\xi^2}\int_0^\infty r\,dr\,\Phi^2_{n+1;1,1}$$
$$+2\pi a\int_{-\infty}^{+\infty}\frac{d\xi}{a^2+\xi^2}\int_0^\infty r\,dr\,\Phi^2_{n,n+1,n+2} \tag{A7}$$

Since the first integral on the right-hand side is unbounded, $P_n^0-P_n^z$ is not a finite quantity.

APPENDIX B

As in Appendix A, we only consider expressions depending on Φ_n. A generalization of (A1) is

$$\int_0^\infty\frac{r^\mu}{(a^2+\xi^2)^{(\mu+1)/2}}\exp\left[-r^2\left(\frac{k}{a-i\xi}+\frac{k'}{a+i\xi}\right)\right]dr=\frac{\lambda_\mu}{g^{(\mu+1)/2}(\xi;\,k,k')}$$
$$g(\xi;\,k,k')=a(k+k')+i\xi(k-k') \tag{B1}$$

and we have proved (Hillion, 1987) that for any integer or half-integer $m>1$ one has

$$\int_{-\infty}^\infty\frac{d\xi}{g^m(\xi;\,k,k')}=\frac{\pi f_m}{(2ak)^{m-1}}\,\delta(k-k')$$
$$f_m=\begin{cases}2/(m-1) & m\text{ integer}\\ 4/(m-1) & m\text{ half-integer}\end{cases} \tag{B2}$$

where $\delta(k-k')$ is the Dirac distribution.

Writing p for pa, one has, according to (27), (27′), (28),

$$\hat{\Phi}_n=\frac{r^n\,e^{-in\varphi}}{(a-i\xi)^{n+1}}\begin{pmatrix}re^{-i\varphi}/(a-i\xi)\\ -i\end{pmatrix}G(p)$$
$$G(p)=\int_0^\infty F(k)\,e^{ikp}\,dk,\qquad p=i\xi+\frac{r^2}{a-i\xi} \tag{B3}$$

and we consider the quantities

$${}_\alpha\Phi^2_{n;\nu_1,\nu_2}=\frac{r^{2n}}{(a^2+\xi^2)^{n+1}}\left(\nu_1+\frac{\nu_2r^2}{a^2+\xi^2}\right)K_\alpha(p),\qquad \alpha=1,2 \tag{B4}$$

with

$$\begin{aligned} K_1(p) &= |G(p)|^2 \\ K_2(p) &= \tfrac{1}{2}[G^*(p)G'(p) + G'^*(p)G(p)] \end{aligned} \tag{B5}$$

$G'(p)$ denotes the derivative of $G(p)$.

According to (16), (18), (B3), (19), (19′) we get

$$\begin{aligned} \hat{P}^r_n &= 4\pi \int_{-\infty}^{+\infty} \xi \, d\xi \int_0^\infty dr \, {}_2\Phi^2_{n+1;1,1} \\ &= 4\pi \int_{-\infty}^{+\infty} \frac{\xi \, d\xi}{(a^2+\xi^2)^{1/2}} \int_0^\infty dr \, (a^2+\xi^2)^{1/2} \, {}_2\Phi^2_{n+1;1,1} \\ \hat{P}^\varphi_n &= 2\pi \int_{-\infty}^{+\infty} d\xi \int_0^\infty dr \, {}_1\Phi^2_{n;n,n+1} \\ &= 2\pi \int_{-\infty}^{+\infty} \frac{d\xi}{(a^2+\xi^2)^{1/2}} \int_0^\infty dr \, (a^2+\xi^2) \, {}_1\Phi^2_{n;n,n+1} \\ \hat{P}^0_n + \hat{P}^\gamma_n &= -2\pi \int_{-\infty}^{+\infty} d\xi \int_0^\infty r \, dr \, {}_2\hat{\Phi}^2_{n+1;1,1} \\ \hat{P}^0_n - \hat{P}^z_n &= 2\pi a \int_{-\infty}^{+\infty} \frac{d\xi}{(a^2+\xi^2)^{1/2}} \int_0^\infty r \, dr \, {}_1\hat{\Phi}^2_{n;n,n+2} \\ &\quad + 2\pi \int_{-\infty}^{+\infty} \left(1 - \frac{2a^2}{a^2+\xi^2}\right) d\xi \int_0^\infty r \, dr \, {}_2\hat{\Phi}^2_{n;1,1} \\ S^z_n &= \pi \int_{-\infty}^{+\infty} d\xi \int_0^\infty r \, dr \, {}_1\hat{\Phi}^2_{n;1,-1} \end{aligned} \tag{B6}$$

Using the Hölder inequality (Korevaar, 1968)

$$\left| \int fg \, d\xi \right| \le \sup|f| \int |g| \, d\xi$$

one has, using the fact that the integrals on r are positive,

$$\begin{aligned} |\hat{P}^r_n| &\le 4\pi \int_{-\infty}^{+\infty} d\xi \, (a^2+\xi^2)^{1/2} \int_0^\infty dr \, {}_1\Phi^2_{n+1;1,1} \\ |\hat{P}^\varphi_a| &\le \frac{2\pi}{a} \int_{-\infty} d\xi \, (a^2+\xi^2)^{1/2} \int_0^\infty dr \, {}_1\Phi^2_{n;n,n+1} \end{aligned} \tag{B7}$$

while, still using the Hölder inequality, $|\hat{P}^0_n - \hat{P}^z_n|$ is finite if the following integrals are bounded:

$$\int_{-\infty}^{+\infty} d\xi \int_0^\infty r \, dr \, {}_1\hat{\Phi}^2_{n;n+1,n+2}, \qquad \int_{-\infty}^{+\infty} d\xi \int_0^\infty r \, dr \, {}_2\Phi^2_{n;1,1} \tag{B8}$$

To sum up, according to (B7) and (B8) and the expressions for $P_n^0 + P_n^z$ and S_n^z, all the quantities (B6) are finite if the integrals $J_{1,2}$ and $J_{2,\alpha}$ are bounded for the values of the parameters α, μ, ν_1, and ν_2 given in Table I:

$$J_{1,\alpha} = \int_{-\infty}^{+\infty} d\xi \int_0^\infty r\, dr\, {}_\alpha\hat{\Phi}^2_{\mu;\nu_1,\nu_2}$$
$$J_{2,\alpha} = \int_{-\infty}^{+\infty} d\xi\, (a^2+\xi^2)^{1/2} \int_0^\infty dr\, {}_\alpha\hat{\Phi}^2_{\mu;\nu_1,\nu_2} \tag{B9}$$

We start with $J_{1,1}$ for $\mu = n$. According to (B3)-(B5) and interchanging the order of the integrations, we get

$$\begin{aligned} J_{1,1} &= \int\int_{-\infty}^{+\infty} dk\, dk'\, F^*(k')F(k) \exp[i\bar{\xi}(k'-k)] \\ &\times \int_{-\infty}^{+\infty} d\xi \int_0^\infty dr \frac{r^{2n+1}}{(a^2+\xi^2)^{n+1}} \left(\nu_1 + \frac{\nu_2 r^2}{a^2+\xi^2}\right) \\ &\times \exp\left[-r^2\left(\frac{k}{a-i\xi} + \frac{k'}{a+i\xi}\right)\right] \\ &= \int\int_{-\infty}^{+\infty} dk\, dk'\, F^*(k')F(k) \exp[i\bar{\xi}(k'-k)] \\ &\times \int_{-\infty}^{+\infty} d\xi \left(\frac{\nu_1\lambda_{2n+1}}{g^{n+1}(\xi;k,k')} + \frac{\nu_2\lambda_{2n+2}}{g^{n+2}(\xi;k,k')}\right) \end{aligned}$$

where we have used (B1). Taking (B2) into account, one finds after integration on k'

$$J_{1,1} = \pi \int_0^\infty dk\, |F(k)|^2 \left(\frac{\nu_1\lambda_{2n+1}f_{n+1}}{(2ak)^n} + \frac{\nu_2\lambda_{2n+2}f_{n+2}}{(2ak)^{n+1}}\right) \tag{B10}$$

Table I

Quantity	Integral	μ	ν_1	ν_2	α	s
$\hat{P}_n^r$	$J_{2,2}$	$n+1$	1	1	2	$n-1/2, n+1/2$
$\hat{P}_n^\varphi$	$J_{2,1}$	n	n	$n+1$	1	$n-3/2, n-1/2$
$\hat{P}_n^0 + \hat{P}_n^z$	$J_{1,2}$	n	1	1	2	$n-1, n$
$\hat{P}_n^0 - \hat{P}_n^z$	$J_{1,1}, J_{1,2}$	n, n	$n+1, 1$	$n+1, 1$	1, 2	$n-1, n, n+1$
$\hat{S}_n^z$	$J_{1,1}$	n	1	-1	1	$n, n+1$

[a]The quantities $\hat{P}_n^\cdot$, $\hat{S}_n^z$ are finite if the integrals Q_s are bounded for $s = n-3/2$, $n-1$, $n-1/2$, n, $n+1/2$, $n+1, \ldots$. If we had used the half-spinor X_m, we would have obtained the same results with m instead of n.

$J_{1,1}$ is bounded if the integrals

$$Q_s = \int_0^\infty \frac{dk}{k^s} |F(k)|^2 \tag{B11}$$

are bounded for $s = n, n+1$.

One has a similar computation for J_{12} with $k|F(k)|^2$ instead of $|F(k)|^2$, so that the integrals Q_s have to be bounded for $s = n-1, n$. Using (B1), one checks easily that J_{21} and J_{22} are deduced from J_{11} and J_{12}, respectively, by changing n into $n-1/2$. In Table I, we give the values of s for which the integrals Q_s must be bounded to make the quantities (B6) finite.

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